LET'S TAKE IT FROM THE TOP

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April 26, 2024

1. Introduction

Have you ever looked at the sky and wonder how stars and planets rotate? How do we understand, model, and predict their motion centuries from now? It turns out, planet motion can be much more complicated than one may perceive, and significant challenges lie around planet motion studies: planets are extremely large, most of them lightyears away, and impossible to experiment with. Nevertheless, we may be able to grasp the fundamentals of these grand questions through investigations of small, everyday objects around us: spinning tops. We aim to provide future researchers an intuitive way to understand *precession, nutation, rotation*, three fundamental rotational motions of planet rotations, through visualization of top motion simulations.

In the AniTop project, we provide numeric simulations of motion of three types of integrable tops, as well as beautiful and detailed visualizations of their motion. We solved quantities that describe top motion by modeling their energy states (Hamiltonian) and establishing equations using constants of motion associated with each type of spinning top. Finally, we provide the open source AniTop MATLAB package that simulates and renders visualizations of these three types of spinning tops, helping future researchers understand spinning tops more intuitively.

2. Scope of Study

2.1. **Model Simplification.** First of all, it is impossible for us to compute the motion of all particles associated with a top, their interactions on a quantum level, or other things happening on a microscopic level. What we really care about is the overall behavior of these spinning tops. To simplify the modeling process, we treat all tops in this project as rigid bodies, where no internal deformation occurs [5, Chapter 3.1]. This allows us to focus on the macro-scale motion of these tops without needing to model internal deformations, decreasing model complexity and improves computational performance. It also means that all quantities of length we mention later associated with a top (L, a) are constants.

Date: April 26, 2024.

We also consider, in this study, only **rotation** of these tops, not their translations. This idealization allows us to consider these tops as ideal (integrable) systems, ones with periodic motion that is analytically solvable, rather than chaotic systems.

2.2. **The Euler, Lagrange, and Kovalevskaya Tops.** In general, rotating tops are chaotic systems that are difficult to predict because of their constantly changing energy states, angular momenta, and many other physical quantities. This means that a small change in initial conditions (for example, slightly higher initial angular velocity) will lead to vastly different behavior after some period of time.

There are, however, three types of tops that have a sufficient number of conserved quantities to be considered integrable: the Euler, Lagrange, and Kovalevskaya Tops.

Definition 2.1. [1] The Euler Top is, physically, a rigid body moving about its center of mass without any net force/torque acting on the body.

Definition 2.2. [6, 3.2.2] The Lagrange Top deals with two moments of inertia that are the same and the center of gravity lies on the symmetry/rotation axis. There is no constraint on the moment of inertia on the third axis.

Definition 2.3. [8, 1] The Kovalevskaya Top has a unique ratio between three moments of inertia, $I_1 = I_2 = 2I_3$, and the center of gravity lies within the plane of two equal moments, but not necessarily on the rotation axis.

These three types of tops, each having more constraint than the previous, resemble planetary motion differently, and opens up opportunities to approximate motion of different types of planets. We will be specifically modeling, solving motions for, and animating these three types of tops in this project.

3. Methods

Before we dive into the mathematics behind our computation, it is imperative to build some background intuition as to why we chose the approach that we did, as well as some background knowledge on physics in order to better understand computing processes presented in section 4.

3.1. **Representing Rotation: Euler Angle.** You may be familiar, or even use the 3D Cartesian coordinate to describe positions of objects in our 3D world with a triplet of numbers (x, y, z). However, how do we represent the "position" of a rotation (or as some call, "orientation")? We may draw some inspiration from the 2D case. In a plane, we specify a rotation by specifying a rotation angle (around the origin). Rotation around the origin in 2D can be thought of as a rotation around a Z-axis that goes through the origin out of the page, perpendicular to the X-Y plane we work with. Thus, we can extend that concept to 3D and define a rotation as a series of three rotations, each around one axis. The set of three angles that we rotate each axis around by is called a set of Euler angles, which we denote as (ϕ, θ, ψ).

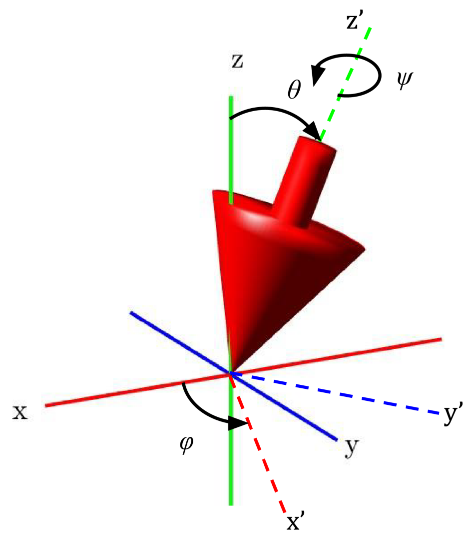

FIGURE 1. Euler Angles

Definition 3.1. [3, 1.1.6] We define

- ϕ as the angle at which the top rotates around the world Z-axis (z),
- θ as the angle at which the top rotates around the body X-axis (x') after the first rotation,
- ψ as the angle at which the top rotates around the body Z-axis (z') after the first two rotations.

The angles ϕ, θ, ψ are sometimes called the angles of *precession, nutation, rotation*, respectively. These quantities are crucial in modeling planet rotations, as planet motion usually includes a combination of them at different rates. For reference, the Earth rotates roughly every 24 hours, nutates every 18.6 years, and precesses every 26,000 years. [2, 8.10]

Definition 3.2. [4, 5.9] We define

- *precession* as the rotation of the body around the fixed Z-axis, where the body Z-axis and fixed Z-axis may not align.
- *nutation* as the variation of angle between the fixed Z-axis and the body Z-axis.
- *rotation* as the rotation of the body around the body Z-axis.

Having sorted out the mathematical representation of rotations, it becomes clear that **we need a series of Euler Angles** from a simulation for us to render an animation of the motion of the tops. We may now proceed to finding mathematical tools that produce us these desired quantities.

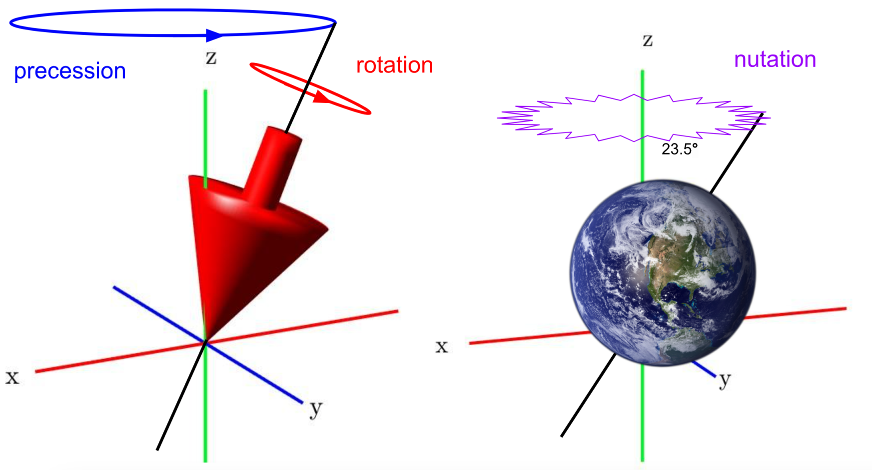

FIGURE 2. Euler Angles

3.2. **Physics of Rotation.** You may be familiar with the Newtonian laws of Physics ($F = ma$, etc). Almost all of these laws for linear motion have corresponding rotational versions. In the case of rotational force (or torque, usually denoted as τ), we define $\tau = I\alpha$, where I is the moment of inertia of the object (corresponding to m, the mass of the object in linear motion), and α is the angular acceleration (corresponding to a, the linear acceleration). We will be extensively using the concept of moment of inertia, demoted I, later in this article.

Definition 3.3. [7, 10.5] The moments of inertia of an object around one rotation axis is defined as

$$I = \int_r \rho(r) \cdot r^2$$

where $\rho(r)$ is the density of the object when it has distance r away from a fixed rotation axis. Like mass M of an object, I is an inherent physical property that remains constant at all times.

We also introduce the concept of velocity in rotations: angular velocity ($\omega = \frac{d}{dt}\theta = \dot{\theta}$), the instantaneous change in angle of a rotating object, as well as generalized momentum for rotation: angular momentum ($P = I\omega$).

Note that there exists a natural coordinate system for all of the objects in 3D. Thus, by considering rotations in 3-dimensional space, we extend the above concepts to 3-dimensional spaces as well, with $I = [I_1, I_2, I_3]^T, \omega = [\omega_1, \omega_2, \omega_3]^T$ (for angular velocities in the world frame, equivalent to Euler angle rates, we denote them specially as $[\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$), and $P = [P_\phi, P_\theta, P_\psi]^T$, associated with 3 axes respectively.

3.3. **Modeling Tops: Hamiltonian Mechanics.** While many of us are familiar with Newtonian Mechanics ($F = ma$, etc), Newtonian Mechanics is actually very difficult when applied to the analysis of rotating systems like these tops. This is because Newtonian Mechanics require a fixed coordinate system (normally a Cartesian coordinate system), and changing coordinate systems will result in extremely cluttered, non-continuous equations that are difficult to solve, if possible at all. Instead, we model the tops motion with Hamiltonian mechanics, building off of the law of conservation of energy. This allows us to use any coordinate system for our quantities, and convert between coordinate systems freely, to produce (relatively more) clean and concise equations.

Definition 3.4. [6, Definition 3.1] A Hamiltonian system is defined by a triple (M, ω, H), where (M, ω) is a $2n$-dimensional sympletic manifold, and H is a smooth function, called *Hamiltonian*. Hamiltonian systems are governed by the following differential equations:

$$-\frac{\partial H}{\partial x} = \dot{p}_x \quad \frac{\partial H}{\partial p_x} = \dot{x}$$

In the case of tops modeling, the Hamiltonian is the energy state of the top, which we can obtain from standard mechanics equations. Using Hamiltonian Mechanics, we are able to carry out derivations and computations shown in section 4, which shows the mathematics behind our simulation methods.

With these mathematical tools, we are now able to start modeling and solving motion profiles for each of these three tops.

4. Top Specific Modeling

4.1. **Euler Top Modeling.** We chose the following approach to solve for the motion of an Euler top (each quantity implicitly being a function of time):

$$H \xrightarrow[\text{Equation}]{\text{Hamilton's}} ODE(\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix}) \xrightarrow[\text{ODE}]{\text{solve}} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \xrightarrow[\text{coordinates}]{\text{change}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \xrightarrow[\text{w.r.t } t]{\text{integrate}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

From elementary physics, you may recall that we define the rotation energy as $T = \frac{1}{2} I\omega^2$ where I is the moment of inertia and ω is the angular velocity. Likewise, this concept can be extended into 3-dimensional physics. Recalling Definition 2.1, we can define the Hamiltonian for the Euler top as:

$$H_{Euler} = T + V = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

where T is the kinetic energy of a 3-dimensional rotating top, and V is the potential energy (which for the Euler top, $V = 0$).

To solve for the motion of Euler tops as integrable systems, we first decrease system complexity by recognizing the constants of motion in an Euler top: the system energy (H) and angular momenta (P_ϕ, P_θ, P_ψ). With these constants, we can proceed to our calculations.

With some sophisticated calculations involving Lie Algebra and some other mathematical tools, we can acquire the following differential equations, a special case of the Euler's Equations of Motion [5, Eq. 36.5]:

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} (I_2 - I_3)I_1 \cdot \omega_2\omega_3 \\ (I_3 - I_1)/I_2 \cdot \omega_1\omega_3 \\ (I_1 - I_2)/I_3 \cdot \omega_1\omega_2 \end{bmatrix}$$

With the help of ordinary differential equation (ODE) numeric integraters, we are able to solve this set of differential equation and obtain a discrete series of angular velocities $[\omega_1(t), \omega_2(t), \omega_3(t)]^T$.

However, since angular velocities are defined with respect to the body axes, in order for our animation to correctly visualize results (with a fixed camera position), we must convert those to rate of rotation in the world frame. We can translate this local rotation to Euler angle rates in the world reference frame with the change of coordinate equation [3, Eq. 2.9]

$$\begin{bmatrix} \dot{\phi}(t) \\ \dot{\theta}(t) \\ \dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} \frac{\sin(\psi)}{\sin(\theta)} & \frac{\cos(\psi)}{\sin(\theta)} & 0 \\ \cos(\theta) & \sin(\psi) & 0 \\ -\frac{\cos(\theta)\sin(\psi)}{\sin(\theta)} & -\frac{\cos(\theta)\cos(\psi)}{\sin(\theta)} & 1 \end{bmatrix} \cdot \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}$$

Finally, we are able to obtain a series of Euler Angles (what we need, as specified in 3.1) by integrating the Euler Angle rates.

$$\begin{bmatrix} \phi(t) \\ \theta(t) \\ \psi(t) \end{bmatrix} = \sum_{i=1}^{T} (\Delta t_i \cdot \begin{bmatrix} \dot{\phi}(t) \\ \dot{\theta}(t) \\ \dot{\psi}(t) \end{bmatrix})$$

Using these set of equations, we are able to obtain a heading of the Euler top at predetermined timestamps for our AniTop animation package, which will be introduced in section 5.

Note that this method involves using Euler's method to numerically integrate the Euler angles. This means that over extended periods of time, the simulation results may show significant deviation from the real-world result, especially if simulated at low frequency. We chose $f = 25$ (25 steps per second, or 0.04 second per step) simulation for a balance between accuracy, performance, and animation rendering considerations. This number is carried over for the other two types of tops as well.

This method inherits a flaw in the flow of computation: when we convert the angular velocities to Euler angle rates, the conversion is susceptible to what is known as the gimbal locking problem: singularity in the change of coordinate matrix. This causes spikes in the solved Euler angle rates, and therefore large deviation from correct results. The common solution amongst the engineering community is using Quaternions, representing rotation with a 4-dimensional quantity. However, our initial exploration with

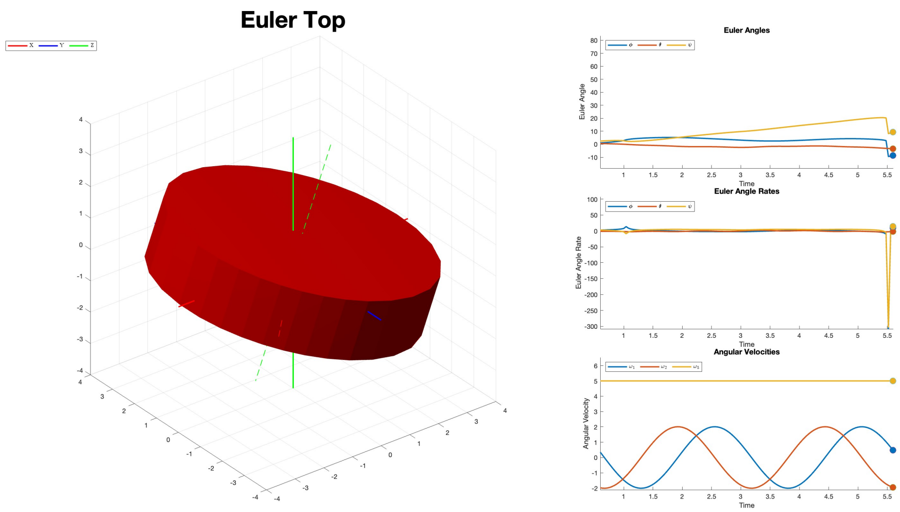

FIGURE 3. Euler Top Animation Snapshot from AniTop

this approach was unsuccessful. Future research into fixing such problem will be conducted to improve the AniTop simulator.

4.2. **Lagrange Top Modeling.** We chose the following approach to solve for the motion of a Lagrange top:

$$H \xrightarrow[\text{Equation}]{\text{Hamilton's}} ODE\left(\begin{bmatrix}\dot{\phi}\\ \ddot{\theta}\\ \dot{\psi}\end{bmatrix}\right) \xrightarrow[\text{1st order}]{\text{convert}} \begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\\ \ddot{\theta}\end{bmatrix} \xrightarrow[\text{ODE}]{\text{solve}} \begin{bmatrix}\phi\\ \theta\\ \psi\\ \dot{\theta}\end{bmatrix} \xrightarrow[\text{Euler angles}]{\text{take}} \begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix}$$

Again, we begin the modeling of Lagrange Top motion with its Hamiltonian.

$$H_{Lag} = T + V = \frac{P_\theta}{2I_1} + \frac{(P_\phi - P_\psi \cos^2(\theta))^2}{2I_1 \sin^2(\theta)} + \frac{P_\phi^2}{2I_3} + MgL\cos(\theta)$$

Note that the potential energy is modeled as $V = MgL\cos(\theta)$, where M is the mass of the top, g is the gravitational constant ($g \approx 9.8m/s^2$ on earth surface), and L is the height of the center of mass of the top, when it is standing perfectly right side up.

To solve for the motion of Lagrange tops as integrable systems, we first decrease system complexity by recognizing the constants of motion in an Euler top: the system energy (H) and angular momenta (P_θ, P_ψ). With these constants, we can proceed to our calculations.

To simplify calculations, we define the following constants:

$$a = \frac{P_\psi}{I_1}, b = \frac{P_\phi}{I_1}, \beta = \frac{2 \cdot MgL}{I_1}$$

Thus, we can obtain the following set of differential equations:

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{b - a\cos(\theta)}{\sin^2(\theta)} \\
\frac{d\psi}{dt} &= a\frac{I_1}{I_3} - \frac{(b - a\cos(\theta))\cos(\theta)}{\sin^2(\theta)} \\
\frac{d^2\theta}{dt^2} &= \frac{(a^2 + b^2)\cos(\theta)}{\sin^3(\theta)} - ab\frac{3 + \cos(2\theta)}{2\sin^3(\theta)} + \frac{\beta}{2}\sin(\theta)
\end{aligned}$$

Observe that the third equation is a second-order differential equation, which our numeric solver could not deal with. We introduce another dummy variable $d\theta$ and dummy differential equation to convert into a pair of first-order differential equations for MATLAB's ODE solver to handle.

$$\begin{cases} \frac{d}{dt}\theta = d\theta \\ \frac{d(d\theta)}{dt} = \frac{(a^2+b^2)\cos(\theta)}{\sin^3(\theta)} - ab\frac{3+\cos(2\theta)}{2\sin^3(\theta)} + \frac{\beta}{2}\sin(\theta) \end{cases}$$

Now that we have four first order differential equations, we are able to solve them numerically and obtain the series of Euler angles $[\phi, \theta, \psi]^T$ that describe the heading of the top through time. This could be directly rendered by our animator to show the motion of this particular Lagrange top.

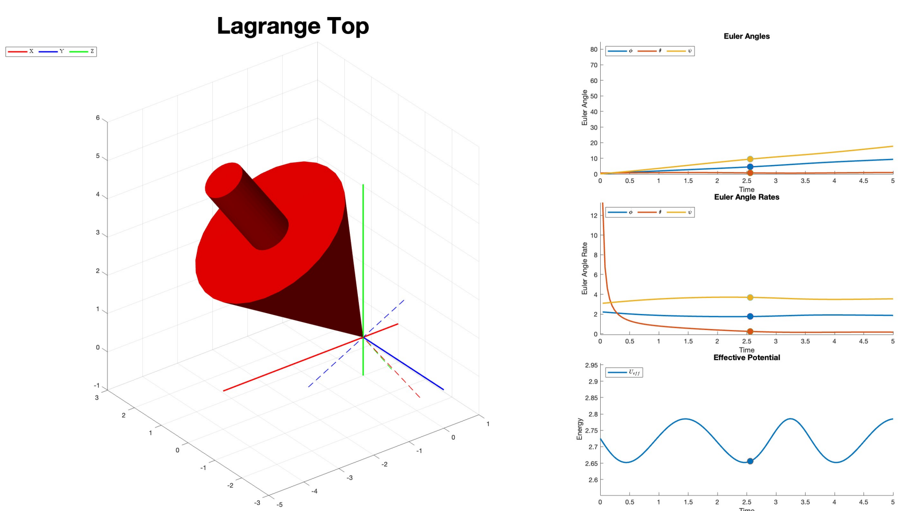

Figure 4. Lagrange Top Animation Snapshot from AniTop

4.3. **Kovalevskaya Top Modeling.** We chose the following approach to solve for the motion of a Lagrange top:

$$H \xrightarrow[\text{Equation}]{\text{Hamilton's}} ODE(\begin{bmatrix} \dot{\phi} \\ \ddot{\theta} \\ \dot{\psi} \\ \dot{P}_\theta \\ \dot{P}_\psi \end{bmatrix}) \xrightarrow[\text{ODE}]{\text{solve}} \begin{bmatrix} \phi \\ \theta \\ \psi \\ P_\theta \\ P_\psi \end{bmatrix} \xrightarrow[\text{Euler angles}]{\text{take}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

First, the Hamiltonian of a Kovalevskaya top can be described as

$$H_{Kov} = \frac{p_\theta^2}{2I} + \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I\sin^2(\theta)} + \frac{p_\psi^2}{4I} + Mga\sin(\theta)\sin(\psi)$$

where a is the distance between the center of mass of the top and the rotation axis. As mentioned in previous sections, the Kovalevskaya top has moments of inertia that follows $I_1 = I_2 = 2I_3$. Thus, we simplify the expressions by setting $I = I_1$. Again, we identify the constants of motion for a Kovalevskaya top: the system energy H, angular momentum P_ϕ, and the Kovalevskaya invariant K, which we will not go into details here. Upon reducing system complexity, we are able obtain the following differential equations:

$$\begin{aligned}
\frac{d\phi}{dt} =& \frac{p_\phi - p_\psi\cos(\theta)}{I\sin^2(\theta)} \\
\frac{d\theta}{dt} =& \frac{p_\theta}{I} \\
\frac{d\psi}{dt} =& -\frac{p_\psi}{2I} - \frac{(p_\theta - p_\psi)\cos(\theta)}{2I\sin^2(\theta)} \\
\frac{dP_\theta}{dt} =& -\frac{(p_\theta - p_\psi\cos(\theta))p_\psi\sin(\theta)}{2I\sin^2(\theta)} + \frac{(p_\phi - p_\psi\cos(\theta))^2}{2I\sin^3(\theta)}\cos(\theta) \\
& - Mga\cos(\theta)\sin(\psi) \\
\frac{dP_\psi}{dt} =& - Mga\sin(\theta)\cos(\psi)
\end{aligned}$$

Solving this set of differential equations numerically, we obtain the series of Euler angles $[\phi, \theta, \psi]^T$ that describe the heading of the Kovalevskaya top through time. The animator can now take this information to render the motion of this particular Kovakevskaya top. Although the solved angular momentum P_θ, P_ψ are not used in the animation, they are vital parts of this set of differential equations, thus required to be solved.

5. The AniTop Package

The AniTop package is designed as an all-rounded and scalable solution to simulating and visualizing tops motion. Using methods described above,

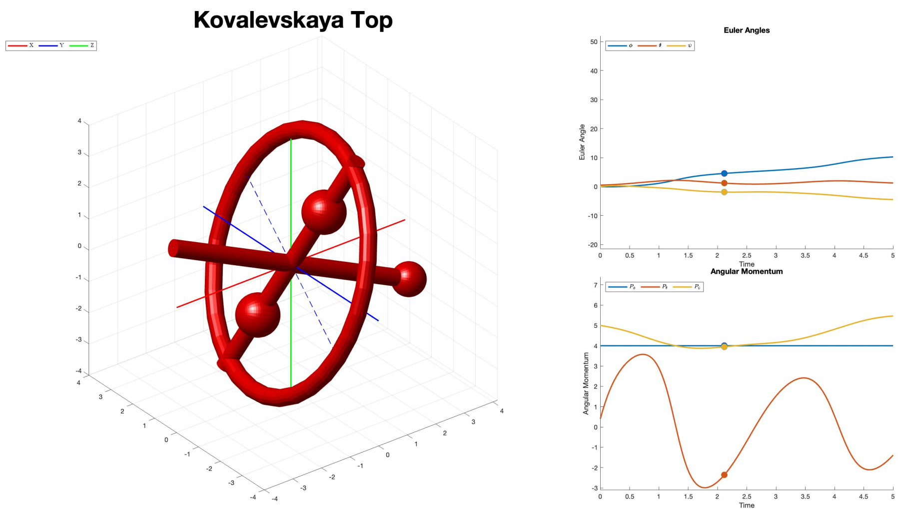

FIGURE 5. Kovalevskaya Top Animation Snapshot from AniTop

we built the simulator portion of the AniTop package. The animation portion is created to smoothly animate the simulation results to reflect changes in observation. Users can experience the amazing graphics of the AniTop program by simply acquiring our program and typing "AniTop" in the MATLAB terminal.

For more advanced uses, AniTop also supports users to set custom initial conditions to different tops, including moments of inertia, initial angular velocities, and initial headings, the three conditions that govern motion of these tops. AniTop also supports batch-simulation and rendering with a single configuration file, where the formatting is intuitive and enables efficient controlled experimentation and visualization. The complete program architecture is presented in Figure 6.

Our animation includes not only 3D graphics, but also 2D plots of the quantities of interest, such as Euler angle rates for the Euler top and angular momentum for the Kovalevskaya top. The variation of these quantities may help explain phenomena observed in top spinning, helping users gain a deeper understanding of these tops under different initial conditions.

For detailed documentation, usage guide, and examples for the AniTop package, please refer to our Gitlab repository. Have fun spinning!

6. CONCLUSION

We present the AniTop package for MATLAB to simulate and animate the motion of three integrable tops: the Euler, Lagrange, and Kovalevskaya tops, computed with analytical results and numeric integrations. We hope this package help the next generation researchers better understand the fundamentals of planet motion in fun, convenient, and insightful ways.

7. Acknowledgements

We sincerely give thank to Dr. Alejandro Bravo-Doddoli and Kausik Das for their guidance and advisory of this project. We also thank Dr. Nir Gadish and the University of Michigan LoG(M) program for making this experience possible. Crucially, I would like to thank my teammates: Haley Gipson, Anna Huang, Ramon Guerra for their wonderful inspirations and contributions to this project.

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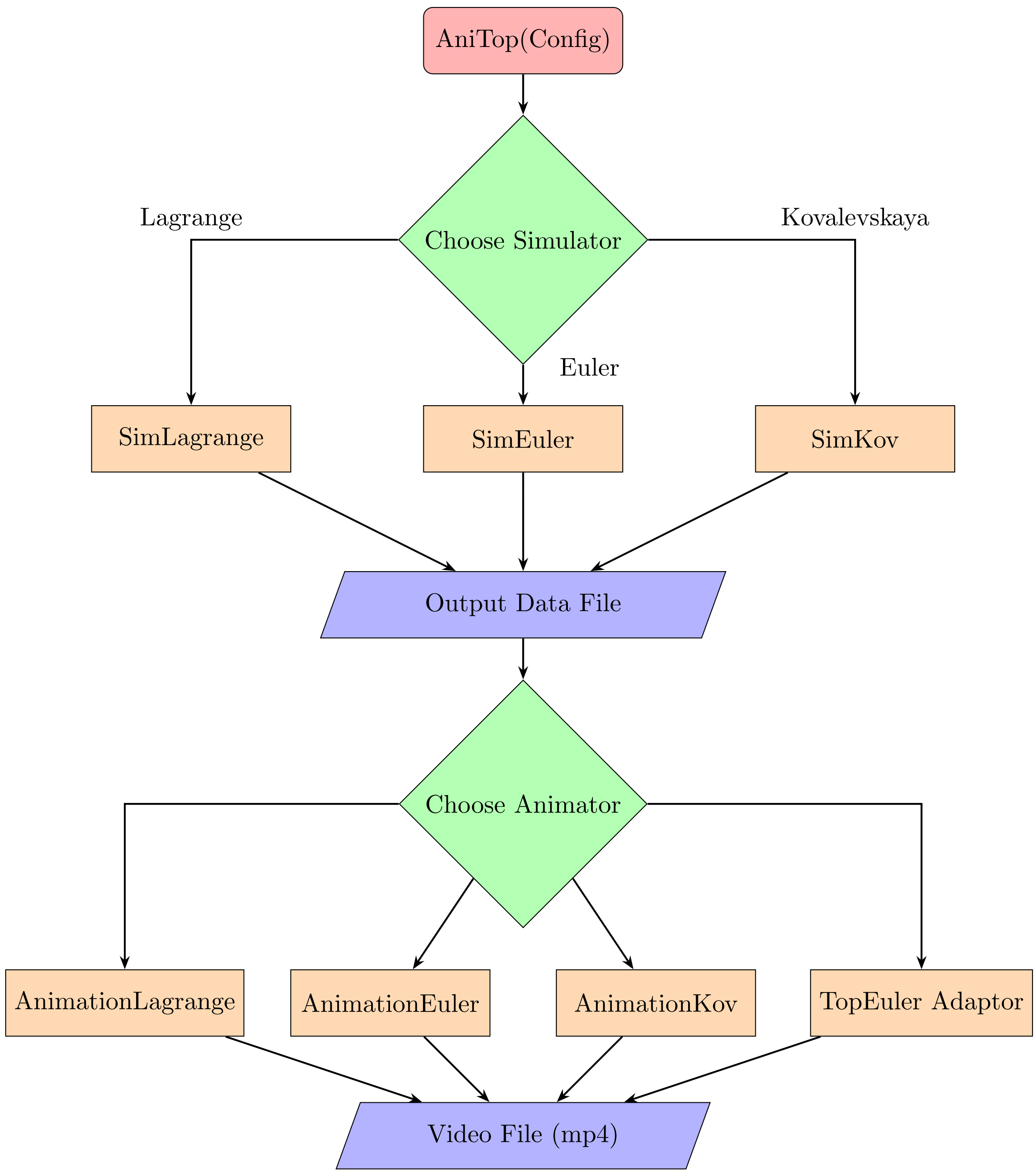

FIGURE 6. AniTop Program Architecture