# Let’s Take It From the Top


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## ABSTRACT


- The study of rigid body motion is equivalent to the study of dynamics in the group of rotations $\mathbb{R}^3$ denoted by $SO(3)$.
- General dynamics in $SO(3)$ are **chaotic.** There exist only three examples where the dynamics are **integrable** (non-chaotic): Lagrange, Euler and Kovalevskaya tops.
- These tops have many potential applications in the study of planetary motion, but they are difficult to visualize.


**OBJECTIVE:**
**Simulate Lagrange, Euler, and Kovalevskaya Tops in MATLAB**

## THEORETICAL BACKGROUND

**RIGID BODY MOTION:** Tops (and also the Earth) can undergo three motions

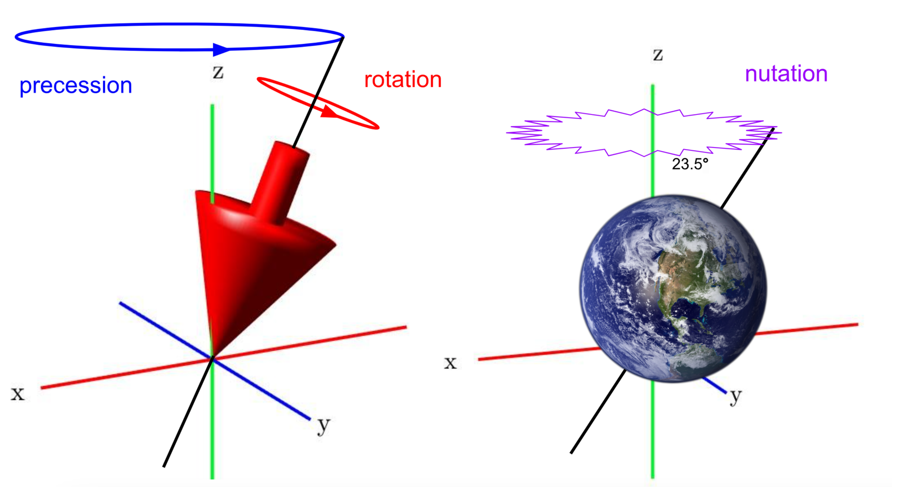



**Figure 1:** *Rotation, precession, and nutation diagram*

### LIE GROUPS

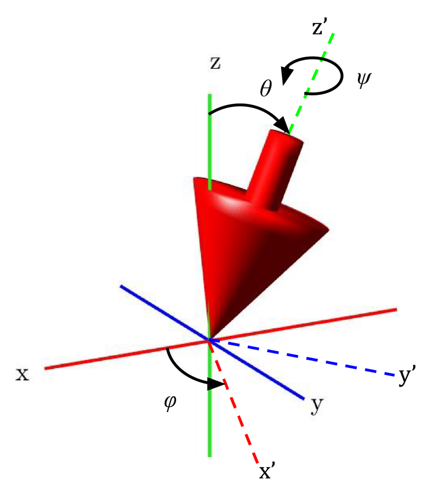
- Lie groups model continuous symmetries, e.g. $SO(3)$ is the group of rotations in $\mathbb{R}^3$.
- The tangent space $T^*SO(3)$ = space of velocities where the Possion bracket is defined.
- Possion bracket $\{F, H\}$ is a tool we can use to find the equations of motion of our tops where $F$ is a smooth function and $H$ is the Hamiltonian.

**HAMILTON’S EQUATIONS OF MOTION** [Hamilton]

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}.$$

where $H(\mathbf{x}, \mathbf{p})$ given $\mathbf{x} = (x_1, ...., x_n)$ is position and $\mathbf{p} = (p_1, ...., p_n)$ is momentum

### EULER ANGLES



**Definition.** *$F$ is a **constant of motion** if its Poisson bracket commutes with $H$. A given $H$ in $T^*SO(3)$ will be **integrable** if it has 3 linearly independent constants of motion that the Poisson bracket commutes between.*

### MOMENTS OF INERTIA

- Moment of inertia $I$ = “rotational mass.”
- There exists a natural orthonormal set of principal axes for a rigid body $I_1$, $I_2$ and $I_3$ that simplify equations of motion.

## TOP ANIMATIONS

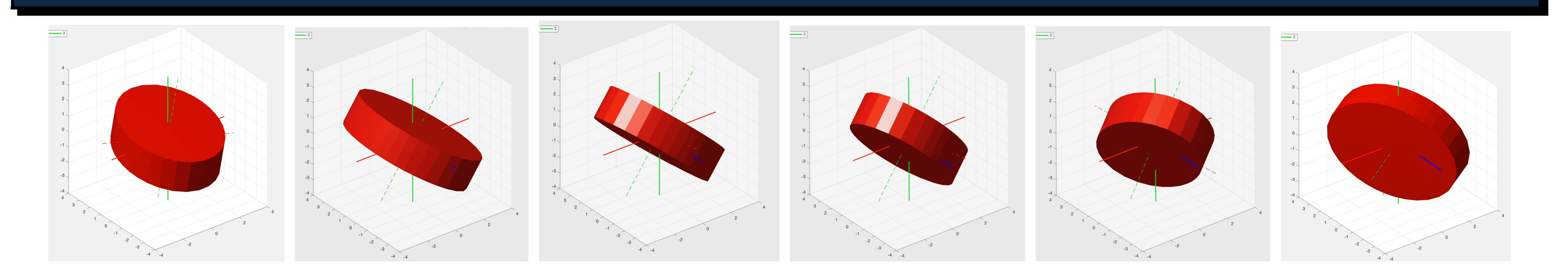


**Figure 2:** *Euler Top Motion Snapshots*

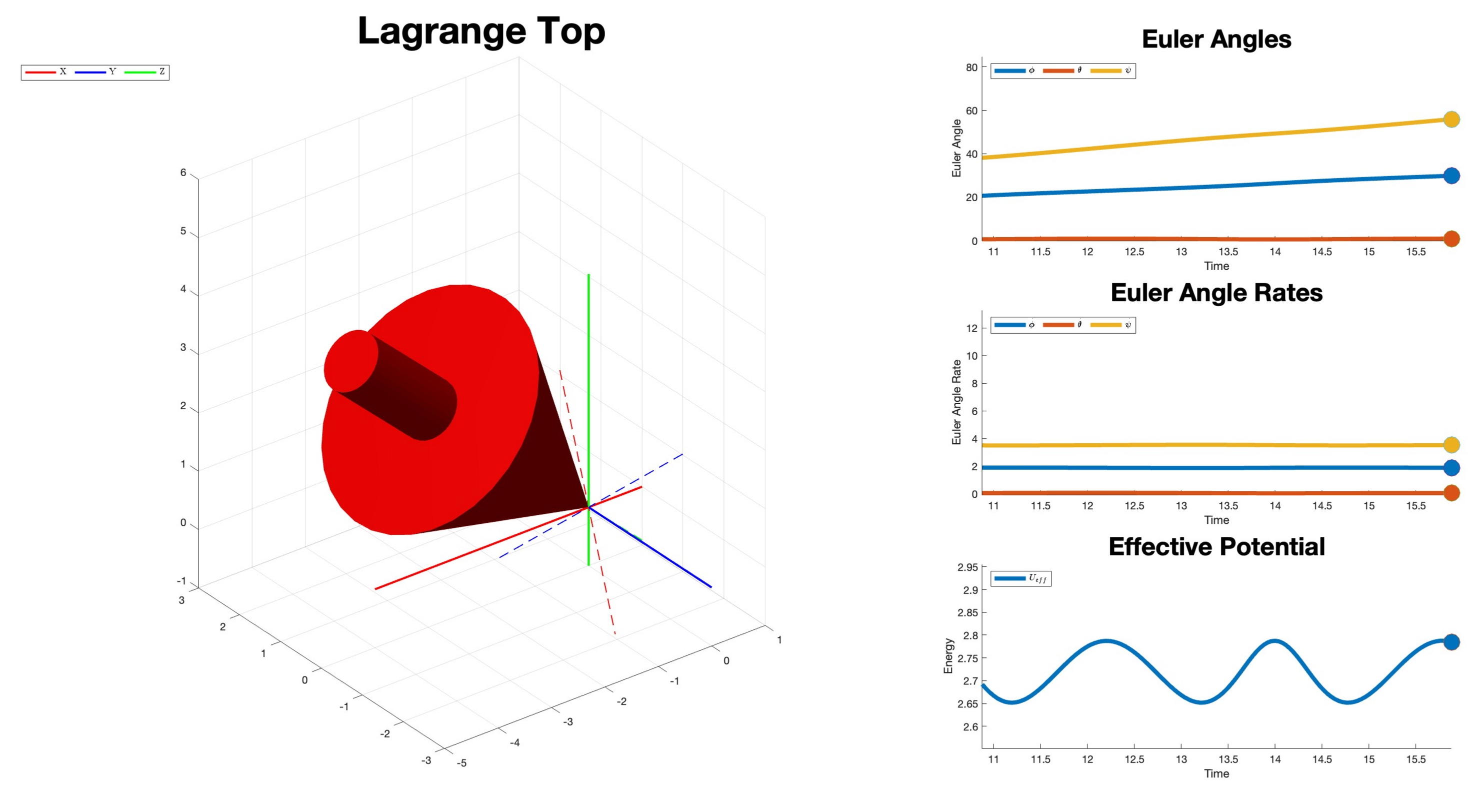



**Figure 3:** *Lagrange Top Animation Interface Snapshot*

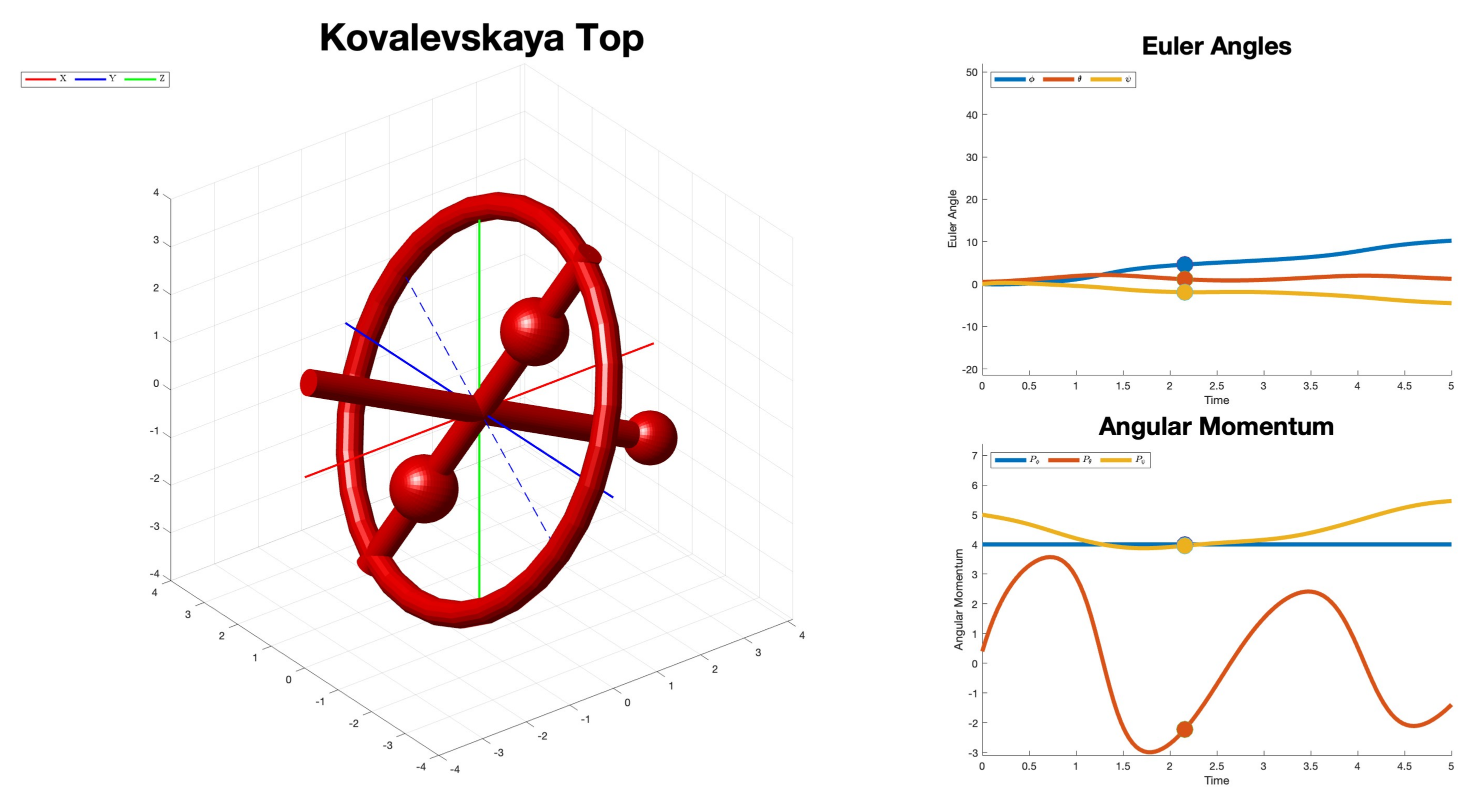



**Figure 4:** *Kovalevskaya Top Animation Interface Snapshot*

| Type | Moments of Inertia | Constants of Motion | Hamiltonian $H$ |
|---|---|---|---|
| **Lagrange** | $I_1 = I_2 < I_3$ | $p_\phi$, $p_\psi$ and $H$ | $\frac{p_\theta^2}{2I_1} + \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1} + \frac{p_\psi^2}{2I_3} + gml\cos(\theta)$ |
| **Euler** | $I_1 < I_2 < I_3$ | $p_1, p_2, p_3$ | $\frac{I_1\omega_1^2}{2} + \frac{I_2\omega_2^2}{2} + \frac{I_3\omega_3^2}{2}$ |
| **Kovalevskaya** | $I_1 = I_2 = 2I_3$ | $p_\phi$, $K$ and $H$ | $\frac{p_\theta^2}{2I} + \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I\sin^2(\theta)} + \frac{p_\psi^2}{4I} + gma\sin(\theta)\sin(\psi)$ |

**Table 1:** *Tops’ Properties Table*

## COMPUTATIONAL METHODS

- We derived differential equations for the Euler angle rates from our Hamiltonians

**Example:** Differential Equations for **Lagrange Top**

$$\frac{d\phi}{dt} = \frac{b - acos(\theta)}{sin^2(\theta)}, \qquad \frac{d\psi}{dt} = a\frac{I_1}{I_2} - \frac{(b - acos(\theta))cos(\theta)}{sin^2(\theta)},$$

$$\frac{d^2\theta}{dt^2} = \frac{(a^2 + b^2)cos(\theta)}{sin^2(\theta)} - ab\frac{3 + cos(2\theta)}{2sin^3(\theta)} + \frac{\beta}{2}sin(\theta)$$

- To find the equations of motion, we used ordinary differential equation solvers and other numerical integration methods.
- Our packages support custom initial conditions including angular velocity, moments of inertia, and initial heading. It also supports configuring multiple tops within one file.

## RESULTS

### EULER TOP VIDEO

- Gimbal locking causes momentary spike in the Euler angle rates.

### LAGRANGE TOP VIDEO

- Nutation causes effective potential $U_{eff}$ to oscillate.

### KOVALEVSKAYA TOP VIDEO

- Back-and-forth motion: a combination of rotation and precession.

## APPLICATIONS

- Integrable tops have many potential applications to planetary motion because planets are chaotic systems, making them difficult to study.
- Earth is similar to a Lagrange top, but its motion is closer to that of a **fast top**, i.e. it rotates much faster than it nutates or precesses.
- For reference, the Earth rotates roughly every 24 hours, nutates every 18.6 years, and precesses every 26,000 years!

**FUTURE RESEARCH:**
**Study Connections Between Motion of Earth and Fast Tops**

## REFERENCES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